NONISOTHERMAL COUETTE FLOW OF A NON-NEWTONIAN FLUID UNDER A PRESSURE GRADIENT

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UDC 532.135

This study is a continuation of [1], where the author qualitatively examined and calculated the nonisothermal Couette flow of a non-Newtonian fluid under the influence of a positive pressure gradient along a plate in the chosen direction of the x axis. It was shown that the phase space of flows of pseudoplastic fluids (flow index $n < 1$) has an integral surface corresponding to flows with a velocity along the plate which is independent of the coordinate y perpendicular to the plate ($v = du/dy = 0$). This corresponds physically to the case of motion of the entire plate–liquid system with a constant velocity (in particular, to a stationary system). It can be shown that the trajectories of this integral surface are singular solutions of the initial system of equations and that the trajectories of the part of the phase space $v < 0$ cross the integral surface into the part of the phase space $v > 0$, i.e., nonisothermal Couette flows with any flow index may have an extreme velocity profile under the influence of a pressure gradient. To prove this, phase coordinates different from those used in [1] are employed.

The system of equations describing nonisothermal Couette flows of a non-Newtonian fluid with a power rheological law under the influence of a pressure gradient has the form

$$\lambda d^2T/dy^2 + \tau^2/\mu = 0; \quad (1)$$

$$\tau = \mu du/dy; \quad (2)$$

$$\mu = \mu_1 |du/dy|^{n-1}; \quad (3)$$

$$\mu_1 = \mu_0 \exp(-\beta T); \quad (4)$$

$$d\tau/dy = dp/dx = A\delta, \quad (5)$$

where λ is the thermal conductivity; T, temperature; τ, shear stress; μ_1, effective viscosity; μ_0, β, and A, constant parameters; $\delta = 1$ for flows in which pressure increases along the coordinate x; $\delta = -1$ for flows in which pressure decreases along the coordinate x; $\delta = 0$ in the absence of a pressure gradient along x.

After introduction of the variable

$$dT/dy = w \quad (6)$$

and allowance for Eqs. (2)-(4), Eq. (1) becomes

$$dw/dy = -|\tau|^{1+\frac{1}{n}} \mu_0^{-1/n} \exp(T\beta/n). \quad (7)$$

Equations (5)-(7) constitute an independent system of differential equations, and its solution can be represented by trajectories in a three-dimensional phase space (w, τ, T). However, due to the relative simplicity of system (5)-(7), its solution can be represented more clearly wihtout loss of generality by trajectories in the plane (w, τ) (Fig. 1, where *a* shows $\delta = 1$ and b shows $\delta = -1$). The intersection of the trajectories with the axis $w = 0$ corresponds to the maximum in the temperature profile. Each of the phase planes contains a line whose interaction with the trajectories is accompanied by a point of inflection in the velocity profile. To find these lines, we will examine the expression for the second derivative of velocity [1]

$$dv/dy = \beta n^{-1} vw + A\delta(\mu_1 n)^{-1} |v|^{-n} v \operatorname{sign} v. \quad (8)$$

Leningrad. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 69-71, March-April, 1986. Original article submitted February 14, 1985.

0021-8944/86/2702-0218$12.50

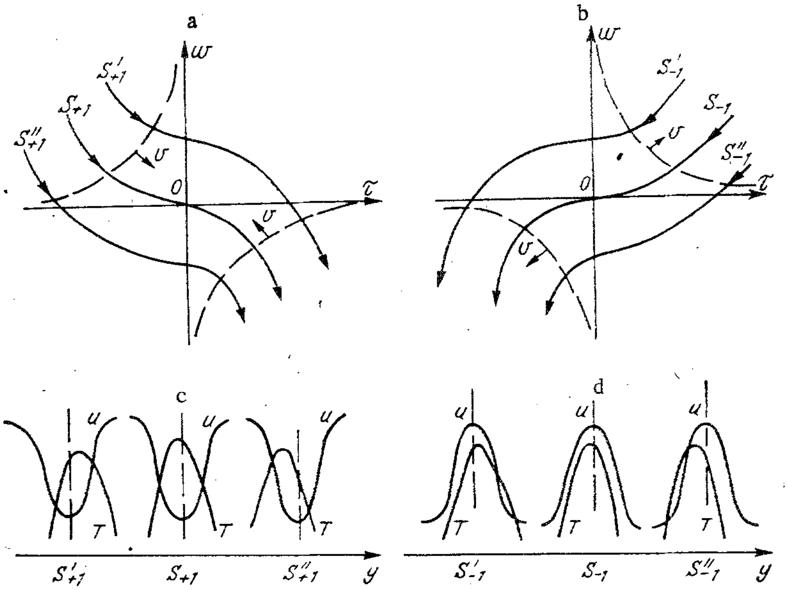

Fig. 1

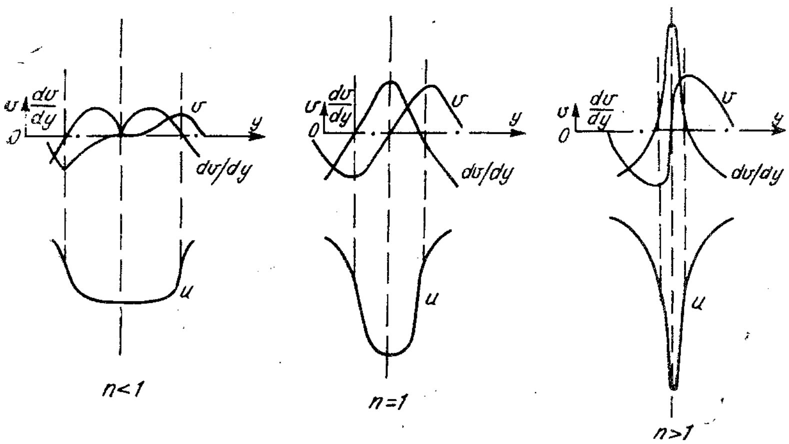

Fig. 2

On the basis of (2) and (3) we express v in the right side of (8) through τ:

$$dv/dy = n^{-1}\mu_0^{-1/n}\left(\beta w\tau|\tau|^{-1+1/n} + A\delta|\tau|^{-1+1/n}\right)\exp(T\beta/n). \quad (9)$$

It follows from (9) that, regardless of the value of n, the derivative dv/dy changes sign, passing through zero, on the lines $w = A\delta/(\tau\beta)$. These lines form hyperbolas located in the second and fourth quadrants of the plane when $\delta = 1$ and in the first and third quadrants when $\delta = -1$. The derivative dv/dy does not change sign on the line $\tau = 0$, while the velocity has an extremum.

Each phase plane contains three types of trajectories — S_{+1}, S_{+1}', S_{+1}'' in the plane $\delta = 1$ and S_{-1}, S_{-1}', S_{-1}'' in the plane $\delta = -1$. Graphs of the change in temperatures and velocities along these trajectories are shown in Fig. 1c, d (part of each trajectory and the corresponding integral curve are realized with certain boundary conditions). The trajectories differ from one another in the relative location of the extrema in the velocity and temperature profiles. In any case, the maximum in the temperature profile is located between the points of inflection in the velocity profile. It is evident from Fig. 1 that besides the symmetry transformation $y \to -y$, $v \to -v$, $w \to -w$ — relative to which system (1)-(5) is invariant, as was shown in [1] — the solutions of its equations possess one other type of symmetry:

$$x \to -x,\ dp/dx \to -dp/dx,\ u \to -u.$$

It follows from system (5)-(7) and the above analysis that the plane $\tau = 0$ is nonintegral in the three-dimensional phase space (w, τ, T) and the trajectories from the half-space $\tau < 0$

($\tau > 0$) cross into the half-space $\tau > 0$ ($\tau < 0$) at $\delta = 1$ ($\delta = -1$). However, $\tau = 0$ only at $v = 0$. This means that crossing of the plane $\tau = 0$ corresponds to the crossing of the plane $v = 0$, considered in [1], by trajectories into the space (w, v, μ_1). However, the plane $v = 0$ is integral in the space (w, v, μ_1) at $n < 1$. This can occur only when the trajectories forming the integral plane $v = 0$ are singular solutions.* Thus, regardless of the values of the index n, the velocity profile in a nonisothermal Couette flow under the influence of a pressure gradient may be extreme. However, the graphs of velocity for flows of pseudoplastic, Newtonian, and dilatant fluids have distinctive features. These features are connected with the fact that the second derivative of velocity with respect to the independent variable has different values at the extremum of the velocity profile. It follows from Eq. (8) that $dv/dy|_{v=0} = A\delta/\mu_1$ is a finite quantity for a flow of a Newtonian fluid, $dv/dy|_{v=0} = 0$ for a pseudoplastic fluid, and $dv/dy|_{v=0} = \infty$ for a dilatant fluid. The velocity profile with an extremum for the dilatant fluid is considerably steeper than for the Newtonian fluid, while the velocity profile of the pseudoplastic fluid is smoother than for the Newtonian fluid (Fig. 2).

In the absence of a pressure gradient ($\delta = 0$), Eqs. (5)-(7) are integrable; their solutions are obtained in [5]. The difference of the index n from unity does not alter the important property of these flows observed in [6] for a Newtonian fluid – coincidence of the coordinate y of the maximum in the temperature profile with the coordinate y of the point of inflection in the velocity profile and the absence of an extremum in the velocity profile.

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*The works familiar to us do not contain any explanation of conventional methods of proving the singularity of solutions in three-dimensional phase spaces (see [2-4], for example).